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Optimality for (h, φ)-multiobjective programming involving generalized type-I functions

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Abstract Based upon Ben-Tal's generalized algebraic operations, new classes of functions, namely (h, φ)-type-I, quasi (h, φ)-type-I, and pseudo (h, φ)-type-I, are defined for a multiobjective programming problem. Sufficient optimality conditions are obtained for a feasible solution to be a Pareto efficient solution for this problem. Some duality results are established by utilizing the above defined classes of functions, considering the concept of a Pareto efficient solution.

Keywords Multi-objective programming · Pareto efficient solution · (h, φ)-type-I functions · Duality

Mathematics Subject Classification Primary 90C29 · 90C46 · Secondary 26B25

1 Introduction

It is well known that the convexity notion plays an vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems. During the past decades, Generalized convex functions received more attention. Various generalizations of convex functions have appeared in literature. More specifically, Hanson and Mond [1] defined the class of type-I functions. With and without differentiability, the type-I functions were extended to some classes of generalized type-I functions by many researchers,

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and sufficient optimality criteria and duality results are obtained for the multiple objective programming problems involving these functions (see [2–9]).

In the literatures [10,11], Ben-Tal introduced certain generalized operations of addition and multiplication. With the help of Ben-Tal's generalized algebraic operations, a meaningful generalization of convex functions is the introduction of (h, φ)-convex functions, which was given by Avriel [10]. Some basic properties of (h, φ)-convex functions are discussed by Ben-tal [11]. Xu and Liu [12,13] established Kuhn-Tucher necessary optimality conditions for (h, φ)-mathematical programming problem. Zhang [14] considered the sufficiency and duality of solutions for nonsmooth (h, φ)-semi-infinite programming.

Our purpose in present paper is to introduce the notions of type-I and generalized type-I functions for a multi-objective differentiable programming problem in the setting of Ben-Tal's generalized algebraic operations. We derive some Karush-Kuhn-Tucker type of sufficient optimality conditions and duality theorems for a Pareto efficient solution to the problem involving the new classes of type-I and generalized type-I functions. This paper is divided into four sections. Section 2 includes preliminaries and related results which will be used in later sections. Sections 3 and 4 are devoted to establishing sufficient conditions of optimality and duality theorems, respectively.

2 Preliminaries and related results

Let R^n be the n-dimensional Euclidean space and $\mathbb{R}$ be the set of all real numbers. Throughout this paper, the following convention for vectors in R^n will be followed:

$$\begin{array}{lll} x > y & \text{if and only if} & x_i > y_i, i = 1, 2, \ldots, n, \\ x \geqq y & \text{if and only if} & x_i \geqslant y_i, i = 1, 2, \ldots, n, \\ x \geq y & \text{if and only if} & x_i \geqslant y_i, i = 1, 2, \ldots, n, \ \text{but } x \neq y. \end{array}$$

Now, let us recall generalized operations of addition and multiplication introduced by Ben-Tal in Ref. [10].

(1) Let h be an n vector-valued continuous function, defined on R^n and possessing an inverse function h^{-1}. Define the h-vector addition of $x, y \in R^n$ as

$$x \oplus y = h^{-1}\big(h(x) + h(y)\big), \tag{2.1}$$

and the h-scalar multiplication of $x \in R^n$ and $\lambda \in \mathbb{R}$ as

$$\lambda \otimes x = h^{-1}\big(\lambda h(x)\big). \tag{2.2}$$

(2) Let φ be real-valued continuous function, defined on $\mathbb{R}$ and possessing an inverse function φ^{-1}. Then the φ-addition of two numbers, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, is given by

$$\alpha[+]\beta = \varphi^{-1}\big(\varphi(\alpha) + \varphi(\beta)\big), \tag{2.3}$$

and the φ-scalar multiplication of $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ as

$$\lambda[\cdot]\alpha = \varphi^{-1}\big(\lambda\varphi(\alpha)\big). \tag{2.4}$$

(3) The (h, φ)-inner product of vector $x, y \in R^n$ is defined as

$$(x^T y)_{h,\varphi} = \varphi^{-1}\big(h(x)^T h(y)\big). \tag{2.5}$$

Denote

$$\bigoplus_{i=1}^{m} x^i = x^1 \oplus x^2 \oplus \cdots \oplus x^m, \quad x^i \in R^n, i = 1, 2, \ldots, m, \tag{2.6}$$

$$\Big[\sum_{i=1}^{m}\Big]\alpha_i = \alpha_1[+]\alpha_2[+]\cdots[+]\alpha_m, \quad \alpha_i \in \mathbb{R}, i = 1, 2, \ldots, m, \tag{2.7}$$

$$x \ominus y = x \oplus (-1) \otimes y, \quad x, y \in R^n, \tag{2.8}$$

$$\alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta), \quad \alpha, \beta \in \mathbb{R}. \tag{2.9}$$

By Ben-Tal's generalized algebraic operations, it is easy to obtain the following conclusions:

$$\varphi(\lambda[\cdot]\alpha) = \lambda\varphi(\alpha), \tag{2.10}$$

$$\alpha[-]\beta = \varphi^{-1}\big(\varphi(\alpha) - \varphi(\beta)\big), \tag{2.11}$$

$$h(\lambda \otimes x) = \lambda h(x). \tag{2.12}$$

Under the above generalized means, (h, φ)-convex functions can be written as

$$\lambda[\cdot]f(x^1)[+](1-\lambda)[\cdot]f(x^2) \geqq f\big(\lambda \otimes x^1 \oplus (1-\lambda) \otimes x^2\big).$$

Avriel [10] introduced the following concept, which plays an important role in our paper.

Definition 2.1 Let f be a real-valued function defined on R^n, denote $\hat{f}(t) = \varphi\big(f(h^{-1}(t))\big)$, $t \in R^n$. For simplicity, write $\hat{f}(t) = \varphi f h^{-1}(t)$. The function f is said to be (h, φ)-differentiable at $x \in R^n$, if $\hat{f}(t)$ is differentiable at $t = h(x)$, and denoted by $\nabla^* f(x) = h^{-1}\big(\nabla \hat{f}(t)\mid_{t=h(x)}\big)$. In addition, It is said that f is (h, φ)-differentiable on $X \subset R^n$ if it is (h, φ)-differentiable at each $x \in X$. A vector-valued function is called (h, φ)-differentiable on $X \subset R^n$ if each of its components is (h, φ)-differentiable at each $x \in X$.

The most important feature of (h, φ)-convex functions is that they are convex transformable. In other words, they can be transformed into convex functions, as we shall see in the next example.

Example 1 Let $f(x) = \log x$. This is a well-known concave function, defined on $C = \{x \in \mathbb{R} : x > 0\}$. However, $\log x$ is (h, φ)-convex with $h(t) = t$ and $\varphi(\alpha) = e^{\alpha}$.

If f is a differentiable at x, then f is (h, φ)-differentiable at x. We obtain this fact by setting h and φ are identity functions, respectively. However, the converse is not true. Let us see next example.

Example 2 Let $f(x) = \sqrt{|x-1|}$ be a function defined on $\mathbb{R}$. It is clear that f is not differentiable at $x = 1$, but f is (h, φ)-differentiable at $x = 1$, where $h(t) = t$, $\varphi(\alpha) = \alpha^3$, $\alpha \in \mathbb{R}$.

We collect some properties of Ben-Tal's generalized algebraic operations and (h, φ)-differentiable functions from literature [12], which will be used in the squeal.

Lemma 2.1 *Suppose that f, f_i are real-valued functions defined on R^n, for $i = 1, 2, \ldots, m$, and (h, φ)-differentiable at $\bar{x} \in R^n$. Then, the following statements hold:*

(1) $\bigoplus_{i=1}^{m} \lambda_i \otimes x^i = h^{-1}\Big(\sum_{i=1}^{m} \lambda_i h(x^i)\Big)$, $x^i \in R^n$, $\lambda_i \in \mathbb{R}$ *for* $i = 1, 2, \ldots, m$.

(2) $\Big[\sum_{i=1}^{m}\Big]\mu_i[\cdot]\alpha_i = \varphi^{-1}\Big(\sum_{i=1}^{m} \mu_i \varphi(\alpha_i)\Big)$, $\mu_i, \alpha_i \in \mathbb{R}$ *for* $i = 1, 2, \ldots, m$.

(3) $\nabla^*(\lambda[\cdot]f(\bar{x})) = \lambda \otimes \nabla^* f(\bar{x})$, *for* $\lambda \in \mathbb{R}$.

(4) $\nabla^*\Big(\Big[\sum_{i=1}^{m}\Big]\lambda_i[\cdot]f_i(\bar{x})\Big) = \bigoplus_{i=1}^{m} \lambda_i \otimes \nabla^* f_i(\bar{x})$, *for* $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

We need more properties of Ben-Tal generalized algebraic operations.

Lemma 2.2 *Let $i = 1, 2, \ldots, m$. The following statements hold:*

(1) $\lambda[\cdot](\mu[\cdot]\alpha) = \mu[\cdot](\lambda[\cdot]\alpha) = \lambda\mu[\cdot]\alpha$, *for* $\lambda, \mu, \alpha \in \mathbb{R}$;

(2) $\lambda[\cdot]\Big[\sum_{i=1}^{m}\Big]\alpha_i = \Big[\sum_{i=1}^{m}\Big]\lambda[\cdot]\alpha_i$, *for* $\lambda, \alpha_i \in \mathbb{R}$;

(3) $\lambda[\cdot](\alpha[-]\beta) = \lambda[\cdot]\alpha[-]\lambda[\cdot]\beta$, *for* $\lambda, \alpha, \beta \in \mathbb{R}$;

(4) $\Big[\sum_{i=1}^{m}\Big](\alpha_i[-]\beta_i) = \Big[\sum_{i=1}^{m}\Big]\alpha_i[-]\Big[\sum_{i=1}^{m}\Big]\beta_i$, *for* $\alpha_i, \beta_i \in \mathbb{R}$;

(5) $\Big(\big(\bigoplus_{i=1}^{m} x_i\big)^T y\Big)_{h,\varphi} = \Big[\sum_{i=1}^{m}\Big](x_i^T y)_{h,\varphi}$, *for* $x_i, y \in R^n$;

(6) $((\lambda \otimes x)^T y)_{h,\varphi} = \lambda[\cdot](x^T y)_{h,\varphi}$, *for* $x, y \in R^n$, $\lambda \in \mathbb{R}$;

(7) $\Big(\big(\bigoplus_{i=1}^{m} \lambda_i \otimes x_i\big)^T y\Big)_{h,\varphi} = \Big[\sum_{i=1}^{m}\Big]\lambda_i[\cdot](x_i^T y)_{h,\varphi}$, *for* $x_i, y \in R^n$, $\lambda_i \in \mathbb{R}$.

Proof

(1). It is easy to obtain this fact from (2.4).

(2). We have

$$\begin{aligned}
\Big[\sum_{i=1}^{m}\Big]\lambda[\cdot]\alpha_i &= \varphi^{-1}\Big(\lambda \sum_{i=1}^{m} \varphi(\alpha_i)\Big) && \text{by Lemma 2.1 (2)}\\
&= \varphi^{-1}\left(\lambda\varphi\left(\varphi^{-1}\left(\sum_{i=1}^{m} \varphi(\alpha_i)\right)\right)\right)\\
&= \lambda[\cdot]\varphi^{-1}\Big(\sum_{i=1}^{m} \varphi(\alpha_i)\Big) && \text{by (2.4)}\\
&= \lambda[\cdot]\Big[\sum_{i=1}^{m}\Big]\alpha_i && \text{by (2.3) and (2.7)}
\end{aligned}$$

(3). We have

$$\begin{aligned}
\lambda[\cdot](\alpha[-]\beta) &= \lambda[\cdot]\big(\alpha[+](-1)[\cdot]\beta\big) && \text{by (2.9)}\\
&= \lambda[\cdot]\alpha[+](-1)[\cdot]\big(\lambda[\cdot]\beta\big) && \text{by (i) and (ii)}\\
&= \lambda[\cdot]\alpha[-]\lambda[\cdot]\beta && \text{by (2.9)}
\end{aligned}$$

(4). We can see that

$$\begin{aligned}
\Big[\sum_{i=1}^{m}\Big](\alpha_i[-]\beta_i) &= \Big[\sum_{i=1}^{m}\Big]\varphi^{-1}\big(\varphi(\alpha_i)-\varphi(\beta_i)\big) && \text{by (2.11)}\\
&= \varphi^{-1}\Big(\sum_{i=1}^{m}\varphi\big(\varphi^{-1}\big(\varphi(\alpha_i)-\varphi(\beta_i)\big)\big)\Big) && \text{by Lemma 2.1 (2)}\\
&= \varphi^{-1}\Big(\sum_{i=1}^{m}\varphi(\alpha_i)-\sum_{i=1}^{m}\varphi(\alpha_i)\Big)\\
&= \varphi^{-1}\Big(\varphi\Big(\varphi^{-1}\Big(\sum_{i=1}^{m}\varphi(\alpha_i)\Big)\Big)-\varphi\Big(\varphi^{-1}\Big(\sum_{i=1}^{m}\varphi(\alpha_i)\Big)\Big)\Big)\\
&= \varphi^{-1}\Big(\varphi\Big(\Big[\sum_{i=1}^{m}\Big]\alpha_i\Big)-\varphi\Big(\Big[\sum_{i=1}^{m}\Big]\beta_i\Big)\Big) && \text{by Lemma 2.1 (2)}\\
&= \Big[\sum_{i=1}^{m}\Big]\alpha_i[-]\Big[\sum_{i=1}^{m}\Big]\beta_i && \text{by (2.11)}
\end{aligned}$$

(5). We can see that

$$\begin{aligned}
\Big(\Big(\bigoplus_{i=1}^{m}x_i\Big)^T y\Big)_{h,\varphi} &= \varphi^{-1}\Big(\Big(h\Big(\bigoplus_{i=1}^{m}x_i\Big)\Big)^T h(y)\Big) && \text{by (2.5)}\\
&= \varphi^{-1}\Big(\sum_{i=1}^{m}h(x_i)^T h(y)\Big) && \text{by Lemma 2.1 (1)}\\
&= \varphi^{-1}\Big(\sum_{i=1}^{m}\varphi\big((x_i^T y)_{h,\varphi}\big)\Big) && \text{by (2.5)}\\
&= \Big[\sum_{i=1}^{m}\Big](x_i^T y)_{h,\varphi} && \text{by Lemma 2.1 (2)}
\end{aligned}$$

(6). We have

$$\begin{aligned}
((\lambda\otimes x)^T y)_{h,\varphi} &= \varphi^{-1}\big(h(\lambda\otimes x)^T h(y)\big) && \text{by (2.5)}\\
&= \varphi^{-1}\big(\lambda h(x)^T h(y)\big) && \text{by (2.12)}\\
&= \varphi^{-1}\big(\lambda\varphi(\varphi^{-1}(h(x)^T h(y)))\big)\\
&= \lambda[\cdot]\varphi^{-1}(h(x)^T h(y)) && \text{by (2.4)}\\
&= \lambda[\cdot](x^T y)_{h,\varphi} && \text{by (2.5)}
\end{aligned}$$

(7). This is a direct consequence of (5) and (6).

Lemma 2.3 *Suppose that function φ, appears in Ben-Tal's generalized algebraic operations, is strictly monotone with $\varphi(0) = 0$. Then, the following statements hold:*

(a) Let $\lambda \geqq 0$, $\lambda, \alpha, \beta \in \mathbb{R}$ and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \leqq \lambda[\cdot]\beta$;
(b) Let $\lambda > 0$, $\lambda, \alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. Then $\lambda[\cdot]\alpha < \lambda[\cdot]\beta$;
(c) Let $\lambda < 0$, $\lambda, \alpha, \beta \in \mathbb{R}$ and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \geqq \lambda[\cdot]\beta$;

(d) *Let* $x, y \in R^m$, *and* $x \leqq y$. *Then* $\left[\sum_{i=1}^{m}\right] x_i \leqq \left[\sum_{i=1}^{m}\right] y_i$;

(e) *Let* $x, y \in R^m$, *and* $x \leq y$. *Then* $\left[\sum_{i=1}^{m}\right] x_i < \left[\sum_{i=1}^{m}\right] y_i$;

Proof We only prove (a) and (e), because the proofs of (b)–(c) and (d) are similar to those of (a) and (e), respectively. Without loss of generality, we suppose that φ is strictly monotone increasing on $\mathbb{R}$.

(a). Since $\lambda \geqq 0$, we have that

$$\begin{aligned} \alpha \leqq \beta &\Rightarrow \varphi(\alpha) \leqq \varphi(\beta) \\ &\Rightarrow \lambda\varphi(\alpha) \leqq \lambda\varphi(\beta) \Rightarrow \varphi^{-1}\big(\lambda\varphi(\alpha)\big) \leqq \varphi^{-1}\big(\lambda\varphi(\beta)\big) = \lambda[\cdot]\alpha \leqq \lambda[\cdot]\beta. \end{aligned}$$

(e). Since $x \leq y$, there exists at least an index k such that

$$\begin{aligned} &x_k < y_k \\ &x_i \leqq y_i \quad \text{for all } \; i \neq k. \end{aligned}$$

Hence

$$\begin{aligned} &\varphi(x_k) < \varphi(y_k) \\ &\varphi(x_i) \leqq \varphi(y_i) \quad \text{for all } \; i \neq k. \end{aligned}$$

Consequently,

$$\sum_{i=1}^{m} \varphi(x_i) < \sum_{i=1}^{m} \varphi(y_i).$$

Since φ is strictly monotone increasing, we get

$$\varphi^{-1}\Big(\sum_{i=1}^{m} \varphi(x_i)\Big) < \varphi^{-1}\Big(\sum_{i=1}^{m} \varphi(y_i)\Big).$$

It yields from Lemma 2.1 (2) that $\left[\sum_{i=1}^{m}\right] x_i < \left[\sum_{i=1}^{m}\right] y_i$.

Lemma 2.4 *Suppose that* φ *is a continuous one-to-one strictly monotone and onto function with* $\varphi(0) = 0$. *Let* $\alpha, \beta \in \mathbb{R}$. *Then* $\alpha < \beta$ *if and only if* $\alpha[-]\beta < 0$.

Proof Without loss of generality, we assume that φ is strictly monotone increasing on $\mathbb{R}$. By the given conditions, we can see that

$$\alpha[-]\beta < 0 \Leftrightarrow \varphi^{-1}\big(\varphi(\alpha) - \varphi(\beta)\big) < \varphi^{-1}(0) \Leftrightarrow \varphi(\alpha) - \varphi(\beta) < 0 \Leftrightarrow \varphi(\alpha) < \varphi(\beta) \Leftrightarrow \alpha < \beta.$$

Throughout of the rest of this paper, we further assume that h is a continuous one-to-one and onto function with $h(0) = 0$. Similarly, suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Under the above assumptions, it is clear that $(0^T x)_{h,\varphi} = (x^T 0)_{h,\varphi} = 0$ for any $x \in R^n$, and $0[\cdot]\alpha = 0$ for any $\alpha \in \mathbb{R}$.

Consider the following multi-objective programming problem:

$$(\text{MOP})_{h,\varphi}\begin{cases}\min & f(x)=(f_1(x),f_2(x),\ldots,f_m(x))^T\\ s.t. & g(x)\leqq 0,\\ & x\in X\subset R^n,\end{cases}$$

where $f:\ X\to R^m$, $g:\ X\to R^p$ are (h,φ)-differentiable. Let $\mathcal{F}$ denote the feasible solutions of $(\text{MOP})_{h,\varphi}$, assumed to be nonempty, that is,

$$\mathcal{F}=\{x\in X:\ \ g(x)\leqq 0\}.$$

□

Definition 2.1 A point $\bar{x}$ is said to be a Pareto efficient solution for $(\text{MOP})_{h,\varphi}$ if $\bar{x}\in\mathcal{F}$ and $f(x)\nleq f(\bar{x})$ for all $x\in\mathcal{F}$.

Definition 2.2 $(f_i,g_j), i=1,2,\ldots,m$ and $j=1,2,\ldots,p$, is said to be (h,φ)-type I with respect to η at $\bar{x}\in X$, if there exists vector-function η: $X\times X\to R^n$ such that for all $x\in X$,

$$f_i(x)[-]f_i(\bar{x})\geqq\Big(\big(\nabla^* f_i(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}, \tag{2.13}$$

and

$$(-1)[\cdot]g_j(\bar{x})\geqq\Big(\big(\nabla^* g_j(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}.$$

Definition 2.3 $(f_i,g_j), i=1,2,\ldots,m$ and $j=1,2,\ldots,p$, is said to be quasi (h,φ)-type I with respect to η at $\bar{x}\in X$, if there exists vector-function η: $X\times X\to R^n$ such that for all $x\in X$,

$$\Big[\sum_{i=1}^m\Big]f_i(x)\leqq\Big[\sum_{i=1}^m\Big]f_i(\bar{x})\ \Rightarrow\ \Big[\sum_{i=1}^m\Big]\Big(\big(\nabla^* f_i(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}\leqq 0,$$

and

$$(-1)[\cdot]\Big[\sum_{j=1}^p\Big]g_j(\bar{x})\leqq 0\ \Rightarrow\ \Big[\sum_{j=1}^p\Big]\Big(\big(\nabla^* g_j(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}\leqq 0.$$

Definition 2.4 (f_i,g_j), $i=1,2,\ldots,m$ and $j=1,2,\ldots,p$, is said to be pseudo (h,φ)-type I with respect to η at $\bar{x}\in X$, if there exists vector-function η: $X\times X\to R^n$ such that for all $x\in X$,

$$\Big[\sum_{i=1}^m\Big]\Big(\big(\nabla^* f_i(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}\geqq 0\ \Rightarrow\ \Big[\sum_{i=1}^m\Big]f_i(x)\geqq\Big[\sum_{i=1}^m\Big]f_i(\bar{x}),$$

and

$$\Big[\sum_{j=1}^p\Big]\Big(\big(\nabla^* g_j(\bar{x})\big)^T\eta(x,\bar{x})\Big)_{h,\varphi}\geqq 0\ \Rightarrow\ (-1)[\cdot]\Big[\sum_{j=1}^p\Big]g_j(\bar{x})\geqq 0.$$

Definition 2.5 (f_i, g_j), $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$, is said to be quasi pseudo (h, φ)-type I with respect to η at $\bar{x} \in X$, if there exists vector-function η: $X \times X \to R^n$ such that for all $x \in X$,

$$\Big[\sum_{i=1}^{m}\Big] f_i(x) \leqq \Big[\sum_{i=1}^{m}\Big] f_i(\bar{x}) \;\Rightarrow\; \Big[\sum_{i=1}^{m}\Big]\Big(\big(\nabla^* f_i(\bar{x})\big)^T \eta(x, \bar{x})\Big)_{h,\varphi} \leqq 0,$$

and

$$\Big[\sum_{j=1}^{p}\Big]\Big(\big(\nabla^* g_j(\bar{x})\big)^T \eta(x, \bar{x})\Big)_{h,\varphi} \geqq 0 \;\Rightarrow\; (-1)[\cdot]\Big[\sum_{j=1}^{p}\Big] g_j(\bar{x}) \geqq 0.$$

Definition 2.6 (f_i, g_j), $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$, is said to be pseudo quasi (h, φ)-type I with respect to η at $\bar{x} \in X$, if there exists vector-function η: $X \times X \to R^n$ such that for all $x \in X$,

$$\Big[\sum_{i=1}^{m}\Big]\Big(\big(\nabla^* f_i(\bar{x})\big)^T \eta(x, \bar{x})\Big)_{h,\varphi} \geqq 0 \;\Rightarrow\; \Big[\sum_{i=1}^{m}\Big] f_i(x) \geqq \Big[\sum_{i=1}^{m}\Big] f_i(\bar{x}),$$

and

$$(-1)[\cdot]\Big[\sum_{j=1}^{p}\Big] g_j(\bar{x}) \leqq 0 \;\Rightarrow\; \Big[\sum_{j=1}^{p}\Big]\Big(\big(\nabla^* g_j(\bar{x})\big)^T \eta(x, \bar{x})\Big)_{h,\varphi} \leqq 0.$$

Let h, φ be identity functions in the above Definition 2.2–2.6, then, (h, φ)-type I, quasi (h, φ)-type I, pseudo (h, φ)-type I, quasi pseudo (h, φ)-type I and pseudo quasi (h, φ)-type I are exactly 'similar' to the concepts of type I, quasi-type I, pseudo-type I, quasipseudo-type I and pesudoquasi-type I (see Definition 2.1–2.5 of Ref. [4]), respectively. Now, we give an example of (h, φ)-type I functions.

Example 3 The functions f: $(0, 1] \to R^2$, $f(x) = \big(f_1(x), f_2(x)\big) = \big(\cos^2(x), -\sin^2(x)\big)$, and $g : (0, 1] \to \mathbb{R}$ defined by $g(x) = \log x$. Let $h(t) = t$ and $\varphi(\alpha) = \arctan(\alpha)$. Then, (f, g) is (h, φ)-type I with respect to $\eta(x, \bar{x}) = 0$ at $\bar{x} = 1$. In fact, observing that $\varphi^{-1}(\alpha) = \tan(\alpha)$ and $h(0) = 0$, $\varphi(0) = \varphi^{-1}(0) = 0$, In this case, we have

$$\begin{aligned} f_1(x)[-]f_1(1) = \cos^2(x)[-]\cos^2(1) &= \tan\big(\arctan(\cos^2(x)) - \arctan(\cos^2(1))\big) \\ &= \frac{\cos^2(x) - \cos^2(1)}{1 + \cos^2(x)\cos^2(1)} \\ &\geqq 0 = \Big(\big(\nabla^* f_1(1)\big)^T 0\Big)_{h,\varphi} \\ &= \Big(\big(\nabla^* f_1(1)\big)^T \eta(x, 1)\Big)_{h,\varphi}, \quad \forall x \in (0, 1], \end{aligned}$$

$$\begin{aligned} f_2(x)[-]f_2(1) = \big(-\sin^2(x)\big)[-]\big(-\sin^2(1)\big) &= \tan\big(\arctan(-\sin^2(x)) - \arctan(-\sin^2(1))\big) \\ &= \frac{-\sin^2(x) - \big(-\sin^2(1)\big)}{1 + \sin^2(x)\sin^2(1)} \\ &\geqq 0 = \Big(\big(\nabla^* f_2(1)\big)^T 0\Big)_{h,\varphi} \\ &= \Big(\big(\nabla^* f_2(1)\big)^T \eta(x, 1)\Big)_{h,\varphi}, \quad \forall x \in (0, 1]. \end{aligned}$$

and

$$(-1)[\cdot]g(1) = \tan\big(-\arctan(\log 1)\big) \geqq 0 = \Big(\big(\nabla^* g(1)\big)^T 0\Big)_{h,\varphi}$$
$$= \Big(\big(\nabla^* g(1)\big)^T \eta(x,1)\Big)_{h,\varphi}, \quad \forall x \in (0,1].$$

By Definition 2.2, we have shown that (f_i, g), $i = 1, 2$, is $(x, \arctan x)$-type I with respect to $\eta = 0$ at $\bar{x} = 1$.

The following example shows that some functions which are not type I at some point can be transformed into (h, φ)-type I.

Example 4 The functions $f\colon (0,\infty) \to R^2$ defined by $f(x) = (f_1(x), f_2(x)) = (|x-1|, \sqrt{|x-1|})$, and $g : (0,\infty) \to \mathbb{R}$ defined by $g(x) = \log x$. It is clearly that (f, g) is not type I with any $\eta(x,\bar{x})$ at $\bar{x} = 1$, because f is not differentiable at $\bar{x} = 1$. However, let $h(t) = t$ and $\varphi(\alpha) = \alpha^3$, then, we can verify that (f, g) is (h,φ)-type I with any $\eta(x,\bar{x})$ at $\bar{x} = 1$. In fact,

$$f_1(x)[-]f_1(1) = |x-1|[-]0 = \big(|x-1|^3 - 0\big)^{\frac{1}{3}}$$
$$\geqq 0 = \Big(0^T \eta(x,1)\Big)_{h,\varphi} = \Big(\big(\nabla^* f_1(1)\big)^T \eta(x,1)\Big)_{h,\varphi},$$
$$\forall x \in (0,\infty).$$

Analogously, we can obtain that f_2 fulfills condition (2.13) at $\bar{x} = 1$. On the other hand, we have

$$(-1)[\cdot]g(1) = \big(-(\log 1)^3\big)^{\frac{1}{3}} = 0 \geqq \Big(0^T \eta(x,1)\Big)_{h,\varphi} = \Big(\big(\nabla^* g(1)\big)^T \eta(x,1)\Big)_{h,\varphi},$$
$$\forall x \in (0,\infty).$$

Thus, it follows from Definition 2.2 that (f_i, g), $i = 1, 2$, is (x, x^3)-type I with respect to any η at $\bar{x} = 1$.

3 Optimality criteria

In this section, we establish Kuhn-Tucker type sufficient optimality conditions for problem $(\mathrm{MOP})_{h,\varphi}$.

Theorem 3.1 *Suppose that there exists a feasible solution $\bar{x}$ for $(\mathrm{MOP})_{h,\varphi}$ and scalars $\bar{\lambda}_i > 0$, $i = 1, 2, \ldots, m$, $\bar{\mu}_j \geqq 0$, $j = 1, 2, \ldots, p$ such that*

$$\Big(\bigoplus_{i=1}^{m} \bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\Big) \oplus \Big(\bigoplus_{j=1}^{p} \bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\Big) = 0, \tag{3.1}$$

and

$$\bar{\mu}_j[\cdot]g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, p. \tag{3.2}$$

If (f_i, g_j) is (h,φ)-type I at $\bar{x}$ with respect to same η, then $\bar{x}$ is a Pareto efficient solution for $(\mathrm{MOP})_{h,\varphi}$.

Proof Since (3.1) holds, by Lemma 2.2(7), for all $x \in X$ we have

$$\left(\left(\bigoplus_{i=1}^{m} \bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\right)^T \eta(x,\bar{x})\right)_{h,\varphi} [+] \left(\left(\bigoplus_{i=1}^{m} \bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\right)^T \eta(x,\bar{x})\right)_{h,\varphi} = 0. \quad (3.3)$$

We proceed by contradiction. Suppose that $\bar{x}$ is not a Pareto efficient solution of $(\mathrm{MOP})_{h,\varphi}$. Then there is a feasible solution $\hat{x}$ of $(\mathrm{MOP})_{h,\varphi}$ and an index k such that

$$f_k(\hat{x}) < f_k(\bar{x}),$$
$$f_i(\hat{x}) \leqq f_i(\bar{x}) \text{ for all } i \neq k.$$

Since $\bar{\lambda}_i > 0$, $i = 1, 2, \ldots, m$, from Lemma 2.3 (a)–(b) we get

$$\bar{\lambda}_k[\cdot]f_k(\hat{x}) < \bar{\lambda}_i[\cdot]f_i(\bar{x}),$$
$$\bar{\lambda}_i[\cdot]f_i(\hat{x}) \leqq \bar{\lambda}_i[\cdot]f_i(\bar{x}) \quad \text{for all } i \neq k.$$

It follows from Lemma 2.3(e) and Lemma 2.4 that

$$Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\hat{x}) < \Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\bar{x}),$$

$$\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\hat{x})[-]\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\bar{x}) < 0. \quad (3.4)$$

By (h,φ)-type-I assumption, for above $\hat{x}$ we have

$$f_i(\hat{x})[-]f_i(\bar{x}) \geqq \left(\left(\nabla^* f_i(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi}, \quad i = 1, 2, \ldots, m,$$

$$(-1)[\cdot]g_j(\bar{x}) \geqq \left(\left(\nabla^* g_j(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi}, \quad j = 1, 2, \ldots, p.$$

Since $\bar{\lambda}_i > 0$, $\bar{\mu}_j \geqq 0$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$, by Lemma 2.3 (a)–(b) and Lemma 2.2 (i), we get

$$\bar{\lambda}_i[\cdot]\big(f_i(\hat{x})[-]f_i(\bar{x})\big) \geqq \bar{\lambda}_i[\cdot]\left(\left(\nabla^* f_i(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi},$$

$$(-1)[\cdot]\big(\bar{\mu}_j[\cdot]g_j(\bar{x})\big) \geqq \bar{\mu}_j[\cdot]\left(\left(\nabla^* g_j(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi}.$$

From Lemma 2.2(3) and (4), and noticing that (3.2) holds, we get

$$\bar{\lambda}_i[\cdot]f_i(\hat{x})[-]\bar{\lambda}_i[\cdot]f_i(\bar{x}) \geqq \left(\left(\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi},$$
$$0 \geqq \left(\left(\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi},$$

By Lemma 2.3 (d), we have

$$\Big[\sum_{i=1}^{m}\Big](\bar{\lambda}_i[\cdot]f_i(\hat{x})[-]\bar{\lambda}_i[\cdot]f_i(\bar{x})) \geqq \Big[\sum_{i=1}^{m}\Big]\left(\left(\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi}, \quad (3.5)$$

$$0 \geqq \Big[\sum_{j=1}^{p}\Big]\left(\left(\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\right)^T \eta(\hat{x},\bar{x})\right)_{h,\varphi}. \quad (3.6)$$

From Lemma 2.2(4) and (3.5), we get

$$\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\hat{x})[-]\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\bar{x}) \geqq \Big[\sum_{i=1}^{m}\Big]\Big(\big(\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi}. \tag{3.7}$$

From (3,7) and (3.6), by Lemma 2.2(7), it yields that

$$\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\hat{x})[-]\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\bar{x}) \geqq \Big(\Big(\bigoplus_{i=1}^{m}\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\Big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi}, \tag{3.8}$$

$$0 \geqq \Big(\Big(\bigoplus_{i=1}^{m}\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\Big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi}. \tag{3.9}$$

From (3.4) and (3.8), we get

$$\Big(\Big(\bigoplus_{i=1}^{m}\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\Big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi} < 0. \tag{3.10}$$

From (3.9) and (3.10), by Lemma 2.3(e), we have

$$\Big(\Big(\bigoplus_{i=1}^{m}\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x})\Big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi}[+]\Big(\Big(\bigoplus_{i=1}^{m}\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\Big)^T \eta(\hat{x},\bar{x})\Big)_{h,\varphi} < 0,$$

which is a contradiction to (3.3).

Theorem 3.2 *Suppose that there exists a feasible solution $\bar{x}$ for (MOP)$_{h,\varphi}$ and scalars $\bar{\lambda}_i > 0$, $i = 1, 2, \ldots, m$, $\bar{\mu}_j \geqq 0$, $j = 1, 2, \ldots, p$ such that* (3.1) *and* (3.2) *hold. If* $(\bar{\lambda}_i[\cdot]f_i, \Big[\sum_{j=1}^{p}\Big]\bar{\mu}_j[\cdot]g_j)$ *is pseudo quasi* (h,φ)*-type I at $\bar{x}$ with respect to same η, then $\bar{x}$ is a Pareto efficient solution for (MOP)$_{h,\varphi}$.*

Proof Since $g(\bar{x}) \leqq 0$ and (3.2) holds, by the pseudo quasi (h,φ)-type I hypothesis on $\Big[\sum_{j=1}^{p}\Big]\bar{\mu}_j[\cdot]g_j$ at $\bar{x}$, for all $x \in X$ we get

$$\Big(\Big(\nabla^*\Big(\Big[\sum_{j=1}^{p}\Big]\bar{\mu}_j[\cdot]g_j(\bar{x})\Big)\Big)^T \eta(x,\bar{x})\Big)_{h,\varphi} \leqq 0.$$

By Lemma 2.1(4), we further get

$$\Big(\Big(\bigoplus_{i=1}^{m}\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\Big)^T \eta(x,\bar{x})\Big)_{h,\varphi} \leqq 0. \tag{3.11}$$

Let $\bar{x}$ not be a Pareto efficient solution for (MOP)$_{h,\varphi}$. Then there exists a feasible $\hat{x}$ for (MOP)$_{h,\varphi}$ and an index k such that

$$\begin{aligned} &f_k(\hat{x}) < f_k(\bar{x}),\\ &f_i(\hat{x}) \leqq f_i(\bar{x}) \ \text{ for all } \ i \neq k. \end{aligned}$$

By the same argument as in that of Theorem 3.1, we get

$$\Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\hat{x}) < \Big[\sum_{i=1}^{m}\Big]\bar{\lambda}_i[\cdot]f_i(\bar{x}).$$

By the pseudo quasi (h,φ)-type I hypothesis on $\bar{\lambda}_i[\cdot]f_i$ at $\bar{x}$, for above $\hat{x}$, we get

$$\Big[\sum_{i=1}^{m}\Big]\Big(\big(\nabla^*(\bar{\lambda}_i[\cdot]f_i(\bar{x})\big)^T\eta(\hat{x},\bar{x})\Big)_{h,\varphi} < 0$$

It follows from Lemma 2.2(7) that

$$\Big(\Big(\bigoplus_{i=1}^{m}\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x})\Big)^T\eta(\hat{x},\bar{x})\Big)_{h,\varphi} < 0. \tag{3.12}$$

By Lemma 2.3(e), it follows from (3.11) and (3.12) that

$$\Big(\Big(\bigoplus_{i=1}^{m}\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x})\Big)^T\eta(\hat{x},\bar{x})\Big)_{h,\varphi}[+]\Big(\Big(\bigoplus_{i=1}^{m}\bar{\mu}_j\otimes\nabla^* g_j(\bar{x})\Big)^T\eta(\hat{x},\bar{x})\Big)_{h,\varphi} < 0.$$

But this is a contradiction to (3.1). The proof is completed.

4 Duality results

Now in relation to $(\mathrm{MOP})_{h,\varphi}$ we consider the following dual problem:

$$(\mathrm{DMOP})_{h,\varphi}\begin{cases}\max\ \ f(y)=(f_1(y),f_2(y),\ldots,f_m(y))^T\\[2ex] s.t.\ \ \Big(\bigoplus_{i=1}^{m}\lambda_i\otimes\nabla^* f_i(y)\Big)\oplus\Big(\bigoplus_{j=1}^{p}\mu_j\otimes\nabla^* g_j(y)\Big)=0\\ \qquad\Big[\sum_{j=1}^{p}\Big]\mu_j[\cdot]g_j(y)\geqq 0,\\ \qquad\lambda>0,\quad\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_m)^T\\ \qquad\mu\geqq 0,\quad\mu=(\mu_1,\mu,\cdots,\mu_p)^T\\ \qquad y\in X.\end{cases}$$

In this section, we provide weak and converse duality relations between problems $(\mathrm{MOP})_{h,\varphi}$ and $(\mathrm{DMOP})_{h,\varphi}$.

Theorem 4.1 *(Weak Duality) Let x and (y,λ,μ) be any feasible solutions for (MOP)$_{h,\varphi}$ and (DMOP)$_{h,\varphi}$, respectively. Let either (I) or (II) below hold:*

(a) (f_i, g_j) is (h,φ)-type I at y with respect to same η;

(b) $(\lambda_i[\cdot]f_i, \Big[\sum_{j=1}^{p}\Big]\mu_j[\cdot]g_j)$ is pseudo quasi (h,φ)-type I at y with respect to same η.

Then

$$f(x)\nleq f(y).$$

Proof Since (y, λ, μ) is feasible solution for $(\mathrm{DMOP})_{h,\varphi}$, by Lemma 2.2(7) and Lemma 2.3 (c), for all $x' \in X$ we have

$$\left(\left(\bigoplus_{i=1}^{m} \lambda_i \otimes \nabla^* f_i(y)\right)^T \eta(x', y)\right)_{h,\varphi} [+] \left(\left(\bigoplus_{i=1}^{m} \mu_j \otimes \nabla^* g_j(y)\right)^T \eta(x', y)\right)_{h,\varphi} = 0, \quad (4.1)$$

and

$$(-1)[\cdot]\left[\sum_{j=1}^{p}\right]\mu_j[\cdot]g_j(y) \leqq 0. \quad (4.2)$$

We proceed by contradiction. Suppose that

$$f(x) \leq f(y).$$

Then, there is exists an index k such that

$$\begin{aligned} f_k(x) &< f_k(y), \\ f_i(x) &\leqq f_i(y) \text{ for all } i \neq k. \end{aligned}$$

By condition (I), with the same argument as that of Theorem 3.1, we can get

$$\left(\left(\bigoplus_{i=1}^{m} \lambda_i \otimes \nabla^* f_i(y)\right)^T \eta(x, y)\right)_{h,\varphi} < 0,$$

and

$$\left(\left(\bigoplus_{i=1}^{m} \mu_j \otimes \nabla^* g_j(y)\right)^T \eta(x, y)\right)_{h,\varphi} \leqq 0.$$

The above two inequalities give

$$\left(\left(\bigoplus_{i=1}^{m} \lambda_i \otimes \nabla^* f_i(y)\right)^T \eta(x, y)\right)_{h,\varphi} [+] \left(\left(\bigoplus_{i=1}^{m} \mu_j \otimes \nabla^* g_j(y)\right)^T \eta(x, y)\right)_{h,\varphi} < 0. \quad (4.3)$$

which contradicts (4.1).

By condition (II), noticing that (4.2) holds, with the similar argument as that of Theorem 3.2, we can get

$$\left(\left(\bigoplus_{i=1}^{m} \mu_j \otimes \nabla^* g_j(y)\right)^T \eta(x, y)\right)_{h,\varphi} \leqq 0,$$

and

$$\left(\left(\bigoplus_{i=1}^{m} \lambda_i \otimes \nabla^* f_i(y)\right)^T \eta(x, y)\right)_{h,\varphi} < 0.$$

The above two inequalities imply (4.3), again a contradiction to (4.1). This completes the proof.

Theorem 4.2 *Suppose that there exist feasible solutions $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for $(MOP)_{h,\varphi}$ and $(DMOP)_{h,\varphi}$, respectively, such that*

$$f_i(\bar{x}) = f_i(\bar{y}), \quad i = 1, 2, \ldots, m. \quad (4.4)$$

Moreover, we assume that the hypotheses of Theorem 4.1 hold at $\bar{y}$, then $\bar{x}$ is a Pareto efficient solution for $(\mathrm{MOP})_{h,\varphi}$.

Proof For any feasible solution x for $(\mathrm{MOP})_{h,\varphi}$, we get from Theorem 4.1 that

$$f(x) \nleq f(\bar{y}). \tag{4.5}$$

Suppose that $\bar{x}$ is not a Pareto efficient solution for $(\mathrm{MOP})_{h,\varphi}$. Then, there exist a feasible solution $\hat{x}$ for $(\mathrm{MOP})_{h,\varphi}$ and an index k such that

$$\begin{aligned} f_k(\hat{x}) &< f_k(\bar{x}), \\ f_i(\hat{x}) &\leqq f_i(\bar{x}) \text{ for all } i \neq k. \end{aligned}$$

Using condition (4.4), we get

$$\begin{aligned} f_k(\hat{x}) &< f_k(\bar{y}), \\ f_i(\hat{x}) &\leqq f_i(\bar{y}) \text{ for all } i \neq k. \end{aligned}$$

This contradicts (4.5). □

Theorem 4.3 *(Converse Duality) Let (y, λ, μ) be a Pareto efficient solution for $(DMOP)_{h,\varphi}$. Moreover, we assume that the hypotheses of Theorem 4.1 hold at y, then y is a Pareto efficient solution for $(MOP)_{h,\varphi}$.*

Proof We proceed by contradiction. Suppose that y is not a Pareto efficient solution for $(\mathrm{MOP})_{h,\varphi}$, that is, there exist $x \in \mathcal{F}$ and an index k such that

$$\begin{aligned} f_k(x) &< f_k(y), \\ f_i(x) &\leqq f_i(y) \text{ for all } i \neq k. \end{aligned}$$

If any one of the hypotheses of Theorem 4.1 holds, it yields in light of Theorem 4.1 that (4.3) is satisfied. This leads to the similar contradiction as in the proof of Theorem 4.1.

5 Conclusions

This paper introduced the concepts of (h, φ)-type I and generalized (h, φ)-type I functions in the setting of Ben-tal' generalized algebraic means, and then, these functions are used to establish some sufficient optimality conditions and dual results for a constrained multiobjective programming. Some researchers have paid attention on mathematical problems under the Ben-tal' generalized algebraic operations, for example: Ben-tal [10] have been applied it to the problems in statistical decision theory, more recently, Xu and Liu [12,13] deal with mathematical programming in the (h, φ)-differentiable case, Zhang [14] and Yuan et al. [15] were concerned on the (h, φ)-generalized directional derivative. Hence, for this purpose, we may conclude that this paper enriched optimization theory in the view of mathematics. Although the results in forms are similar to those of literatures [1,2,4], there are some differences in applications. Let us see the following example:

Example 5 Considering the following multi-objective programming:

$$(\mathrm{P1}) \quad \begin{cases} \min \ f(x) = \left(|x-1|, \sqrt{|x-1|}\right)^T \\ \quad s.t. \ \ g(x) = \log x \leqq 0 \\ \qquad\quad x \in (0, \infty). \end{cases}$$

It is obviously that $\bar{x} = 1$ is the optimal solution (is also the Pareto efficient solution) of (P1). Since f is not differentiable at $\bar{x} = 1$, (f, g) is not type I or generalized type I with respect any η at $\bar{x} = 1$. Thus, some conclusions in the literatures [1,2,4] are not suitable for problem (P1), for example: Theorem 2.3 of Ref. [1], Theorem 4.1 of Ref. [2] and Theorem 3.1 of Ref. [4]. However, in the setting of Ben-tal' generalized algebraic operations, the above mentioned problem can be solved by using suitable h and φ. For example: Let $h(t) = t$ and $\varphi(\alpha) = \alpha^3$, then we can verify that (f, g) is (h, φ)-type I with respect to any η at $\bar{x} = 1$, and for all $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) > 0$ and $\bar{\mu} \geqq 0$, the conditions (3.1) and (3.2) are fulfilled. Consequently, Theorem 3.1 in present paper can be used to problem (P1).

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